Analysis of task A-B

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The solution is hidden in the name of the task. Let us prove its validity by proving the existence of the value $A - B$.

Claim: If a and b are arbitrary integers (positive, negative or zero), then $a - b$ exists and is an integer.

We start from the Peano axioms for $(\mathbb{N}, 0, S)$ and from the recursive definition of addition: $a + 0 = a$ and $a + S(b) = S(a + b)$. By induction one proves associativity, commutativity and cancellativity, that is, that $a + c = b + c$ implies $a = b$. Plus enough set theory to form $\mathbb{N} \times \mathbb{N}$ and a quotient set by a relation.

On $\mathbb{N} \times \mathbb{N}$ we define $(a, b) \sim (c, d)$ exactly when $a + d = c + b$. The pair (a, b) encodes $a - b$, and the condition is a rewriting of $a - b = c - d$ without subtraction. That this is an equivalence relation is verified directly: reflexivity and symmetry are trivial, and for transitivity, if $a + d = c + b$ and $c + f = e + d$, we add f to the first one and obtain $a + d + f = c + b + f$, we substitute $c + f$ with $e + d$ and reach $(a + f) + d = (e + b) + d$, from which by cancellativity $a + f = e + b$. We define $\mathbb{Z} := (\mathbb{N} \times \mathbb{N})/\sim$ and denote the classes by $[a, b]$.

Now, on $\mathbb{Z}$, we define the two operations:

$$[a, b] + [c, d] := [a + c,\ b + d] \qquad \text{and} \qquad [a, b] - [c, d] := [a + d,\ b + c].$$

The correctness is verified as follows: if $[a, b] = [a', b']$ and $[c, d] = [c', d']$, that is, $a + b' = a' + b$ and $c + d' = c' + d$, we add the two equalities termwise and after rearranging we obtain exactly $a + d + b' + c' = a' + d' + b + c$, which by definition says that $(a + d,\ b + c) \sim (a' + d',\ b' + c')$. Hence the difference does not depend on the choice of representatives.

From here on the claim for integers is almost immediate. Let x and y be arbitrary elements of $\mathbb{Z}$. By the definition of the quotient set there exist naturals a, b, c, d with $x = [a, b]$ and $y = [c, d]$. Then $a + d$ and $b + c$ are natural numbers, because addition is a total operation on $\mathbb{N}$ — this is a direct consequence of its recursive definition and induction. Therefore the ordered pair $(a + d,\ b + c)$ is an element of $\mathbb{N} \times \mathbb{N}$, and its class $[a + d,\ b + c]$ is an element of $\mathbb{Z}$. This class is exactly $x - y$. It exists, it is in $\mathbb{Z}$, and by the check above it does not depend on the choice of a, b, c, d. Hence $x - y$ exists for all x, y in $\mathbb{Z}$.

Uniqueness also comes for free. If z is such an element of $\mathbb{Z}$ that $z + y = x$, then by definition $z = x - y$. Indeed, if $z' + y = x$ as well, then $z + y = z' + y$, and it remains to see that addition in $\mathbb{Z}$ is cancellative. We write $z = [p, q]$, $z' = [r, s]$, $y = [c, d]$. From $[p + c,\ q + d] = [r + c,\ s + d]$ it follows that $p + c + s + d = r + c + q + d$, and by cancellativity in $\mathbb{N}$ (applied twice, once for c and once for d) we obtain $p+s=r+q$, that is, $z=z'$. Besides that

$$[a+d,\ b+c]+[c,d]=[a+d+c,\ b+c+d]=[a,b],$$

because $(a+d+c)+b=a+(b+c+d)$, so the element we found really does solve the equation.

Therefore for integers a and b the difference $a-b$ exists, is unique and is an integer. Totality is built into the very construction of $\mathbb{Z}$: subtraction reduces to two additions in $\mathbb{N}$, and addition in $\mathbb{N}$ is total by induction. ■